Scientific Visualization, 2025, volume 17, number 5, pages 10 - 18, DOI: 10.26583/sv.17.5.02

Principles of Organization of Continuity in Discrete Geometrized Space

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Abstract

The paper considers the principle of analytical transition to a local function at points on the domain of an implicit function defining a geometric object. Herewith, a transition to partial derivatives is provided to obtain a general form of an implicit local function describing the local geometry for any single point in the object domain. On the analogy of R-functional modeling, a mathematical apparatus for union/intersecting local geometric characteristics of a local function at a single point is provided to construct a discrete region of a complex geometric object.

An example of the intersection of two functions on a defined domain of arguments demonstrates the obtaining of a discretely geometrized three-dimensional manifold for describing a cylinder.

The proposed work is the continued development of method of the Functional Voxel Modeling which offers an analytical structure for the discrete-continuous description of complex geometric objects instead of the means of linear approximation currently used in this method.

Keywords: domain of definition of the function, Functional Voxel Method (FV-method), Functional Voxel modeling, partial derivatives, local geometric characteristics, local function, R-functional modeling, discrete-continuous domain.

Problem Statement

The idea of analytical modeling of complex geometric objects by implicit functions (as an inverse problem of analytical geometry) has existed for a long time [1-5] and has a number of significant advantages over surface models described by a parametric form for obtaining the coordinates of boundary points.

First of all, in implicit modeling a geometric object is defined by a zero boundary range of values, which adds to the applicability of such approach in engineering calculations. Secondly, it can be noted that there is no limit for the specified dimensionality of space.

However, further application possibilities in computer technologies require an analytical function given by a complicated expression to be additionally simplified by discretization on a given domain. First of all, computer information is discrete and requires a transition from functional continuity to discrete continuity. Discrete continuity should provide properties and identity retention of functional continuity, i.e. it should also be represented by a local function of an implicit form at a specified point.

Research papers [6-10] reflect the basic principles of Functional Voxel modeling method (FV-method), which is based on constructing a local function with an angular metric at each point $(x_1, x_2, \dots, x_{n-1})$ on a given domain:

$$cos\alpha_1 x_1 + cos\alpha_2 x_2 + \cdots + cos\alpha_{n-1} x_{n-1} + cos\alpha_n = 0. \quad (1)$$

The computer storage of angular metrics, referred to in the FV-method as “local geometric characteristics”, is carried out through converting the colour palette into numerical values, which enables to obtain special images of local geometric characteristics (M-images) on a given domain, as well as to reassemble them by inverse transformation into cosines to obtain a local function at the considered point of the M-image.

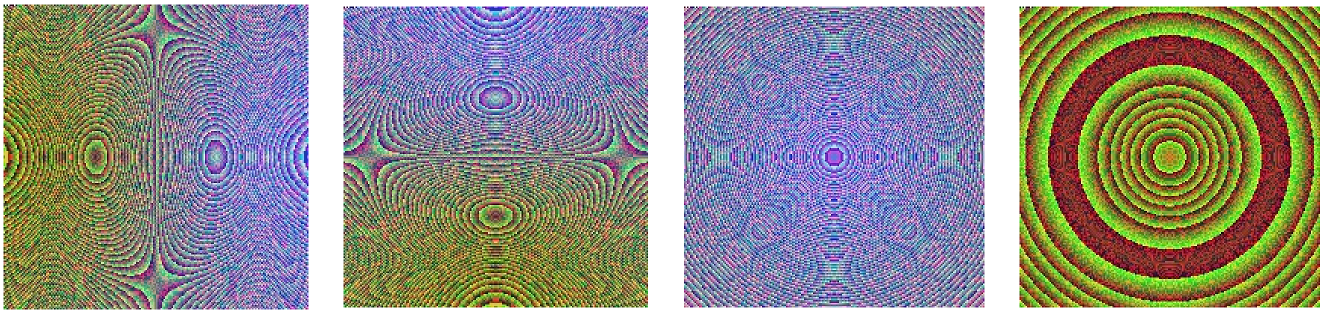
For example, M-images for a circle:

$$z = r^2 - x^2 - y^2 \text{ or } \; r^2 - x^2 - y^2 - z = 0 \tag{2}$$

could be represented by a local function

$$cos\alpha x + cos\beta y + cos\gamma z + cos\delta = 0.$$

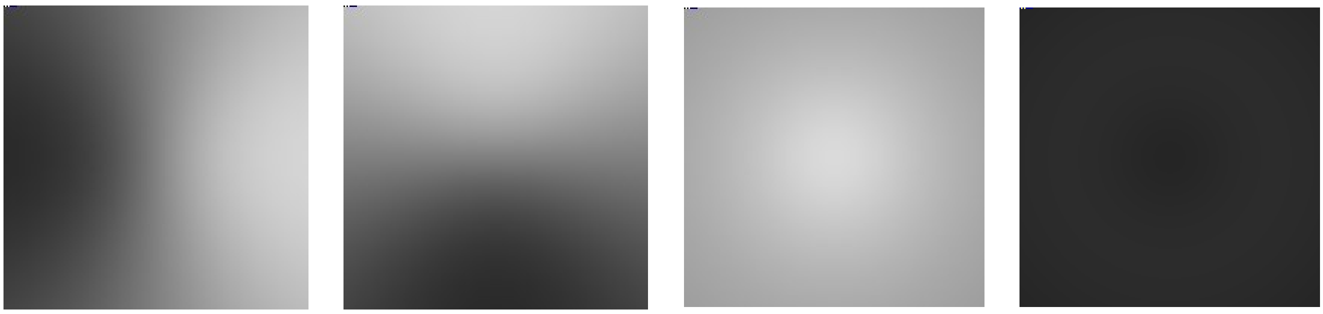
M-images describing the region $x = [-r; r], y = [-r; r]$ are shown in Figure 1.



M_1 M_2 M_3 M_4

Fig 1. M-images of a circle in RGB colour format, which displays 16777215 shades of colour

Figure 2 shows semi-tone images of local geometric characteristics for convenient visual assessment, but with a limited accuracy of representation up to 255 shades of gray.



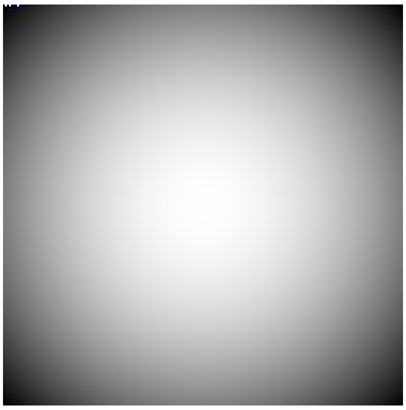
M_1 M_2 M_3 M_4

Fig 2. M-images of a circle. Monochromatic colour palette displaying 256 shades of gray colour

The main advantage of a local function is that it can be quite simply represented as a sum of arguments multiplied by local geometric characteristics, which makes it easy to express the desired variable through the remaining arguments of a function, providing simple calculations for a point selected on a given domain. For example:

$$z = -\frac{cos\alpha}{cos\gamma}x - \frac{cos\beta}{cos\gamma}y - \frac{cos\delta}{cos\gamma}. \tag{3}$$

Figure 3 illustrates in monochrome the region representing values calculated for variable z. Figure 3. a) shows the normalized values of variable z, and Figure 3.b) highlights in blue color the negative values of z outside the object, and positive values inside the object in white.


a)

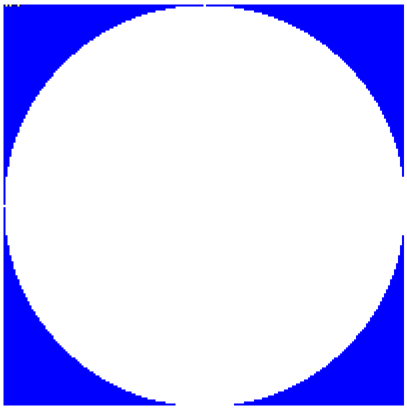

b)

Fig. 3. Values of z distributed over a given region for the function (2): a) values are normalized from white to black b) region separation

Many computer applications based on the Functional-Voxel modelling method (FV-method) have been implemented to solve problems in various areas of mathematical modeling [3-7], resulting in a visual solution that is unlimited in dimension.

At the same time, the proposed FV-method is based on a discrete representation of data, referred to as M-images, obtained by a linear approximation of three nearest neighbour vertices in a regular mesh, which obviously leads to limitations in the accuracy of data representation, as well as to a limited area for calculations.

1. Differential representation of a local function

In order to overcome these limitations, we apply the differential principles of constructing a local function.

Let us consider again an example of a local function defined by an implicit equation and describing a two-dimensional domain of the function on the xOy plane, and z representing its value:

$$cos\alpha x + cos\beta y + cos\gamma z + cos\delta = 0. \tag{4}$$

Let's reduce equation (4) to the differential form:

$$-\frac{cos\alpha}{cos\gamma}x - \frac{cos\beta}{cos\gamma}y - \frac{cos\delta}{cos\gamma} = \frac{cos\gamma}{cos\gamma}z. \tag{5}$$

In essence, formula (5) leads to the following expression:

$$-\frac{\partial z}{\partial x}x - \frac{\partial z}{\partial y}y - \frac{\partial z}{\partial t} = \frac{\partial z}{\partial z}z. \tag{6}$$

which implies that $\partial z/\partial x$, $\partial z/\partial y$, $\partial z/\partial t$ – analytically defined surfaces of partial derivative functions, where $\partial z/\partial z$ equals one.

Let's look at an example with an analytical description of a circle, describing how a discrete local function defined for each point of a given domain can be expanded to a general analytical form.

We consider equation (2) as the initial equation for describing a circle. Taking first order partial derivatives with respect to three arguments, we obtain:

$$\frac{\partial z}{\partial x} = -2x, \qquad \frac{\partial z}{\partial y} = -2y, \qquad \frac{\partial z}{\partial z} = 1. \tag{7}$$

Based on the equality (6), the fourth derivative can be expressed:

$$\frac{\partial z}{\partial t} = -\frac{\partial z}{\partial x}x - \frac{\partial z}{\partial y}y + \frac{\partial z}{\partial z}z = +2x \cdot x + 2y \cdot y + (r^2 - x^2 - y^2). \tag{8}$$

We obtain local geometric characteristics by multiplying all four differential components (derivatives) by the third component $-\partial z/\partial z$

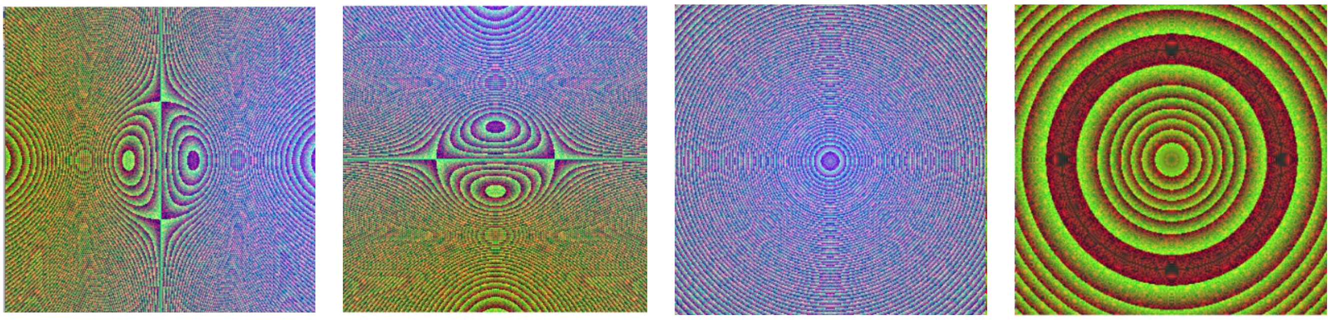
$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial z}x + \frac{\partial z}{\partial y}\frac{\partial z}{\partial z}y + \frac{\partial z}{\partial z}\frac{\partial z}{\partial z}z + \frac{\partial z}{\partial t}\frac{\partial z}{\partial z} = 0 \tag{9}$$

and then, normalizing by the length of the homogeneous normal vector, we get the cosine values for the components:

$$N = \sqrt{1 + \left(\frac{\partial z}{\partial x}\frac{\partial z}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial y}\frac{\partial z}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial t}\frac{\partial z}{\partial z}\right)^2}$$
$$cos\alpha = \frac{\frac{\partial z}{\partial x}\frac{\partial z}{\partial z}}{N}, cos\beta = \frac{\frac{\partial z}{\partial y}\frac{\partial z}{\partial z}}{N}, cos\gamma = \frac{1}{N}, cos\delta = \frac{\frac{\partial z}{\partial t}\frac{\partial z}{\partial z}}{N}. \tag{10}$$

Figure 4 represents M-images displaying local geometric characteristics distributed over the same region for the equation of a circle as those shown in Figure 1, but obtained without linear approximation [6-10].

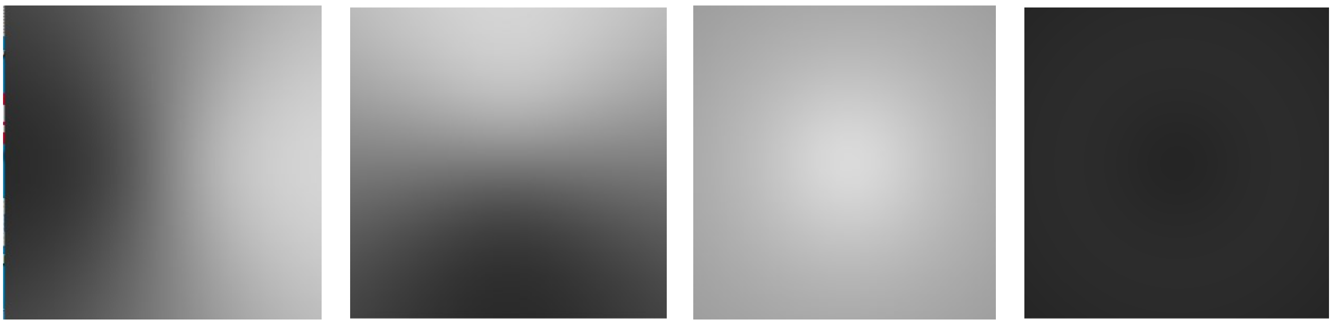
If we compare both images, we see some similarities but with the obvious difference in color patterns. This is due to the increased accuracy of the obtained representation compared to the mesh approximation.



M_1 M_2 M_3 M_4

Fig. 4. M-images of a circle in RGB colour format, which displays 16777215 shades of colour

Note that Figure 5 is quite comparable to Figure 2, since a black-and-white palette and only 256 tone gradations were used to construct the image, i.e. a lower colour resolution.



M_1 M_2 M_3 M_4

Fig. 5. M-images of a circle equation in monochrome format, displaying 256 shades of colour

Given the fact that most analytical functions describing objects of analytical geometry can be differentiable at all points of the domain and along each axis, we can say with confidence that the proposed approach is fully consistent.

This representation significantly expands the scope of application of the local function with all its remarkable properties.

First, we can unequivocally assert an increase in the accuracy of local geometric characteristics. Second, here we have the advantage of the most compact description of local geometric characteristics by analytical expressions, which eliminates the problem of limitations arising from a predefined domain of arguments and the number of gradations of the color palette, which expands its applicability in multi-object modeling.

Of course, it can be argued that, for example, the R-function is difficult to differentiate and the transition to representing a geometry by a local function may, on the contrary, further complicate calculations. Let's try to solve this problem using the developed tools of local geometry, where R-function is adapted to the calculation of local characteristics [10].

2. Example of geometrization of a region for a cylinder

To begin with, using the experience gained in modeling a local function for a circle, let's R-functionally model a three-dimensional cylinder figure with a height of $2h$ as a local function in a three-dimensional region:

$$cos\alpha x + cos\beta y + cos\gamma z + cos\delta u + cos\tau = 0. \tag{11}$$

To do this, we use the intersection of the regions of functions describing two spatial figures u_1 and u_2: an infinite horizontal line with a height equal to $2h$ and an infinite vertical cylinder with the radius r.

$$u_1 = h^2 - z^2 \text{ and } u_2 = r^2 - x^2 - y^2. \tag{12}$$

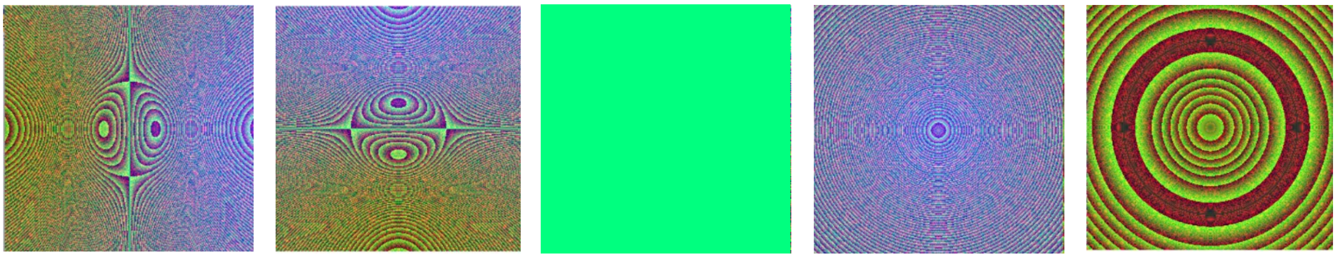
To obtain an analytically described implicit function for a cylinder, an R-intersection is usually used:

$$u = u_1 + u_2 - \sqrt{u_1^2 + u_2^2}. \tag{13}$$

Using partial derivatives, the figure of an infinite cylinder is described by analogy with the equation of a circle, where there is no influence of the z-coordinate:

$$\frac{\partial u}{\partial x} = -2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial u}{\partial u} = 1, \quad \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x}x - \frac{\partial u}{\partial y}y + \frac{\partial u}{\partial u}u, \tag{14}$$
$$u = r^2 - x^2 - y^2.$$

Figures 6 and 7 demonstrate M-images for the region $x = [-r; r], y = [-r; r], z = 0$.



M_1 M_2 M_3 M_4 M_5

Fig.6. M-images of an infinite cylinder in RGB colour format, which displays 16777215 shades of colour

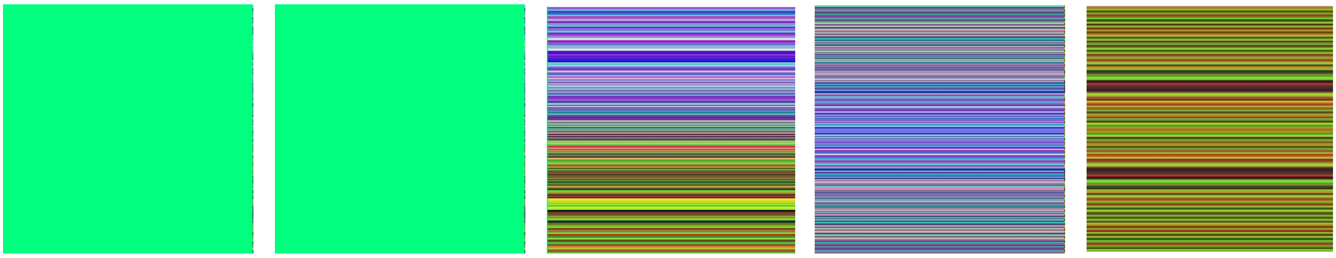
M_1 M_2 M_3 M_4 M_5

Fig.7. M-images of an infinite cylinder in monochrome format, displaying 256 shades of colour

Restrictions along the Oz-axis with the height of $2h$:

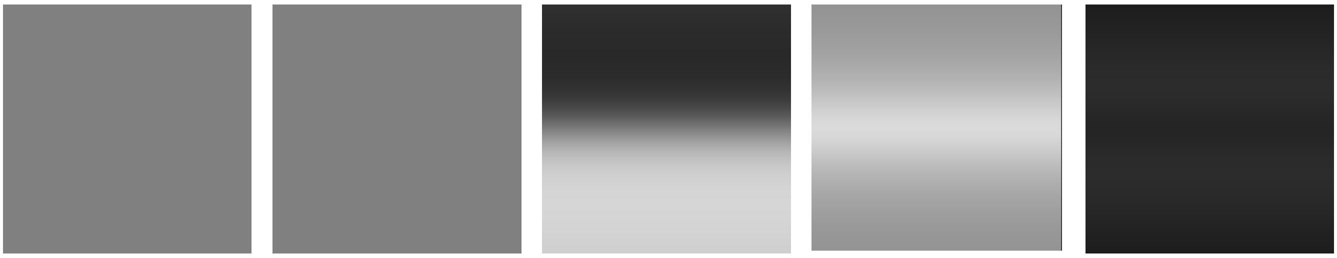
$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = -2z, \quad \frac{\partial u}{\partial u} = 1, \quad \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial z}z + \frac{\partial u}{\partial u}u, \quad u = h^2 - z^2. \tag{15}$$

Figures 8 and 9 demonstrate M-images for this example, the region: $x = [-2h; 2h], y = 0, z = [-2h; 2h]$.



M_1 M_2 M_3 M_4 M_5

Fig.8. M-images of a horizontal line function in RGB colour format, which displays 16777215 shades of colour



M_1 M_2 M_3 M_4 M_5

Fig.9. M-images of a horizontal line function in monochrome format, displaying 256 shades of colour

Let us apply the formula given in [8] for the R-intersection of differentials. For convenience, we will apply the previously defined notations, shortening the differential equation, and transmit it into local geometric characteristics:

$$N = \sqrt{1 + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial z}\frac{\partial u}{\partial u}\right)^2 + \left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial u}\right)^2},$$

$$cos\alpha_1 = \frac{\frac{\partial u}{\partial x}\frac{\partial u}{\partial u}}{N}, cos\alpha_2 = \frac{\frac{\partial u}{\partial y}\frac{\partial u}{\partial u}}{N}, cos\alpha_3 = \frac{\frac{\partial u}{\partial z}\frac{\partial u}{\partial u}}{N}, \quad cos\alpha_4 = \frac{1}{N}, \quad cos\alpha_5 = \frac{\frac{\partial u}{\partial t}\frac{\partial u}{\partial u}}{N}. \tag{16}$$

$$l_1^{(1)} = \frac{cos^{(1)}\alpha_1}{cos^{(1)}\alpha_4}, l_2^{(1)} = \frac{cos^{(1)}\alpha_2}{cos^{(1)}\alpha_4}, l_3^{(1)} = \frac{cos^{(1)}\alpha_3}{cos^{(1)}\alpha_4}, l_5^{(1)} = \frac{cos^{(1)}\alpha_5}{cos^{(1)}\alpha_4},$$

$$l_1^{(2)} = \frac{cos^{(2)}\alpha_1}{cos^{(2)}\alpha_4}, l_2^{(2)} = \frac{cos^{(2)}\alpha_2}{cos^{(2)}\alpha_4}, l_3^{(2)} = \frac{cos^{(2)}\alpha_3}{cos^{(2)}\alpha_4}, l_5^{(2)} = \frac{cos^{(2)}\alpha_5}{cos^{(2)}\alpha_4}, \tag{17}$$

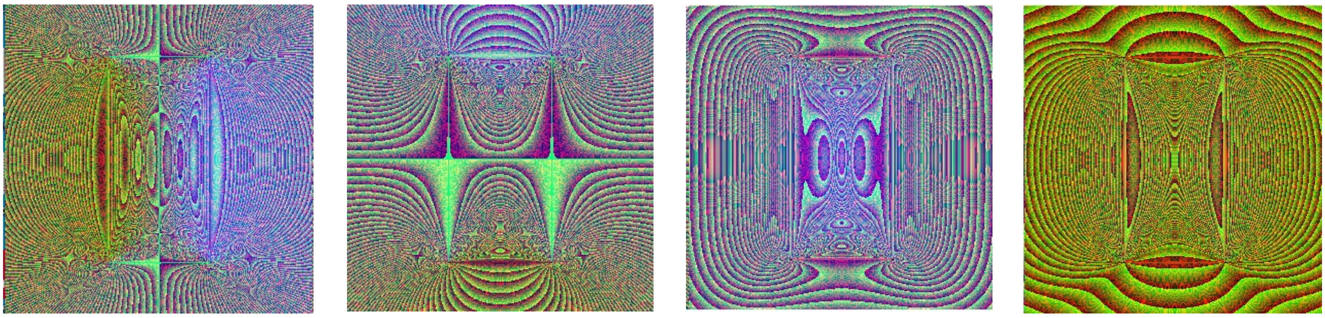
$$u^{(1)} = -l_1^{(1)}x - l_2^{(1)}y - l_3^{(1)}z - l_5^{(1)},$$

$$u^{(2)} = -l_1^{(2)}x - l_2^{(2)}y - l_3^{(2)}z - l_5^{(2)}.$$

$$l_i^{(Cyl)} = l_i^{(1)} + l_i^{(2)} - \left(\frac{l_i^{(1)}u^{(1)} + l_i^{(2)}z^{(2)}}{\sqrt{(u^{(1)})^2 + (u^{(2)})^2}}\right), \; l_4^{(1,2)} = 1, \tag{18}$$

$$i = 1,2,3,5.$$

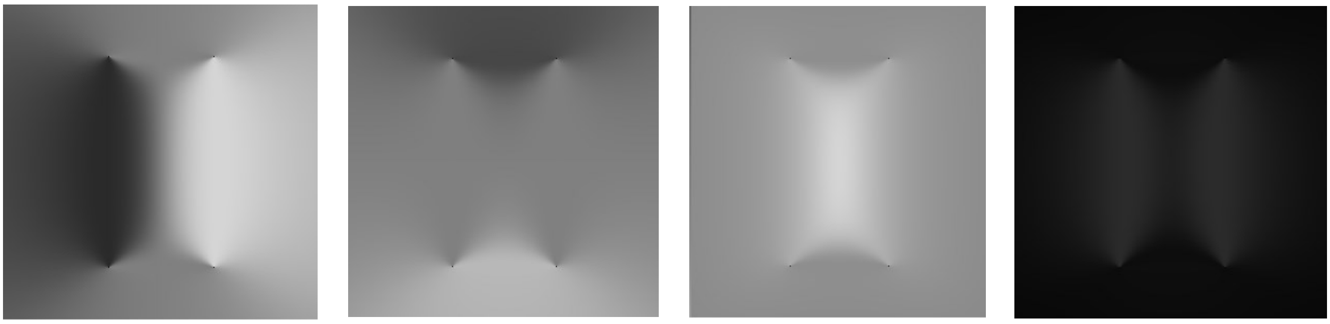
M-images representing a plane section of a cylinder in the xOz-plane in RGB color format for computer storage are shown in Figure 10.



M_1 M_2 M_3 M_4

Fig.10. M-images of a cylindric section in RGB colour format, which displays 16777215 shades of colour

For convenient visual evaluation by a person, Figure 11 shows these M-images in monochrome format.



M_1 M_2 M_3 M_4

Fig.11. M-images of a cylindric section in monochrome format, displaying 256 shades of colour

Conclusions

The presented principle of discretization of continuous space by continuous local functions allows us to formulate further problems related to the analytical description of the geometry of a complex technogenic environment by implicit functions. The idea is to identify a group of basic differentials that at the local level enables description of the geometry of the function space of any dimension and complexity of the formulated object, which allows us to think about expanding the tools of the modern graphics kernel, leading them to a multidimensional representation for the implementation of design and control tasks.

Acknowledgements

The research was carried out within the framework of the scientific program of the National Center for Physics and Mathematics, direction No. 9 "Artificial intelligence and big data in technical, industrial, natural and social systems".

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